

Political Choices in One Dimension: Theory*

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*I am deeply indebted to my many collaborators over the years whose work is cited here and whose insights and analyses have shaped my understanding of the spatial model of politics. But above all my debt is to the pioneering and seminal work of Duncan Black and Anthony Downs. This essay is forthcoming in S. Winer, S. Voigt and B. Grofman (eds.) *Oxford Handbook of Public Choice*.

ABSTRACT

We focus on majority rule processes for choice of a single alternative from a set of alternatives we can regard as locations in one dimensional space (a line), and where the voters who must choose among these alternatives can themselves be viewed as having a most preferred location on this line, with utility for the voter falling off with distance from each voter's "ideal point." We also consider alternatives to this *proximity model* of voting, such as the *directional model*, that emphasizes change in direction vis-a-vis the location of the status quo, and the *mixed model*, that combines directional and proximity components. We briefly review some of the main theoretical results on majority rule voting, such as the *median voter theorem*, and on *value restriction* and other conditions that avoid cycles. We also very briefly consider some other types of models that can be studied in the context of choice of alternatives on a line, such as *threshold models*, and models that voters are making *judgements* rather than expressing preferences. In the companion chapter to this one we consider applications of the spatial model in the context of one dimensional politics in areas such as legislative voting, party competition, and coalition formation.

I. Introduction

There are many situations, both governmental and private, where choices are made by a small (or not so small) group of people who are picking outcome from a set of alternatives, and where each voter can be characterized as having a most preferred outcome from this set and a preference among pairs (or subsets) of alternatives. In many such situations these alternatives can be characterized as points in some multidimensional issue or policy space. We will refer to decisions made in such contexts as involving *spatial voting*. In most such models, voters are identified by what has been called their *ideal point*, a.k.a., *bliss point*, i.e., the position in the multidimensional space that the voter most prefers.

How can we predict what choice (what are the most likely choices) the group as a whole will make in *spatial voting games*? We cannot answer this question without filling in additional details, of which the six most important are: (a) the total number of alternatives, (b) the number of alternatives that will be in the final outcome set, (c) the rule/institutional mechanism by which choices will be made, (d) the number of voters, (e) the nature of voter information and preference structure, and (f) the dimensionality of the issue or policy space. In the discussion that immediately follows we will limit ourselves with respect to each of these parameters.

First, re the number of alternatives, we will simply assume that number to be finite, but pay particular attention to the case where there are either exactly two alternatives, or the agenda is structured by a series of pairwise choices.

Second, we will limit ourselves to the special case where voters are picking a single outcome.¹

¹ In particular we will avoid the complications caused by voters choosing a subset of alternatives or a ranking of alternatives rather than a single alternative (see e.g., Regenwetter et al. 2006; Young, 1986). In our discussion of applications, we will, however, briefly discuss the complications caused by legal choices made by high level courts which go beyond merely reaching a decision about a case as to which party prevails but also specify a rule (a precedent) that influences outcomes in future cases.

Third, we will limit ourselves to the case where decisions are made by majority (or plurality) rule.

Fourth, to simplify the exposition we will limit ourselves to strict preferences and to an odd number of voters, so that we will not need to deal with the complications caused by ties.

Fifth, to further keep the modeling simple, we posit common knowledge by the voters of the location of the alternatives on the various dimensions.²

Finally, while we will reference choices made over multiple policy or issue dimensions, we focus virtually all of our attention on *unidimensional politics*.³

One additional important modeling question involves a choice between a deterministic and a probabilistic approach to voter decision-making.

In *deterministic voting*, in picking a single choice from among some set of alternatives, each voter always prefers that alternative which is closer (in spatial terms) to the voter's own most preferred outcome. This formulation is often referred to as the *proximity* model of voting. In formal terms, denoting the ideal point of the *i*th voters *i*, and using *d* to denote distance with *A* and *B* two distinct alternatives, we have

A is strictly preferred to *B* by voter *i* if and only if $d(B, i) > d(A, i)$

There is also a *probabilistic form of proximity voting*. In the probabilistic version, in a choice between any two alternatives, *A* and *B*, each voter chooses *A* with a certain probability, with that probability a function of the relative closeness of the two alternatives to the voter's bliss point. For the *i*th voter, whose bliss point we also denote *i*, perhaps the simplest probabilistic version of proximity voting specifies that

$$p_i(A \text{ chosen over } B) = d(B, i) / (d(A, i) + d(B, i)).^4$$

² Later we briefly discuss the realism of this assumption and what happens when it is violated.

³The *locus classicus* of work in a single dimension involving party competition is Downs (1957), For a discussion of voting and coalitions in two or more dimensions see Merrill and Grofman (1999), Adams, Merrill, and Grofman (2005) and the essays by Adams and by Schofield in this Handbook.

Even though probabilistic models fit better to empirical observations about voter choice for reasons of simplicity of exposition and mathematical tractability of the models, in this essay, we largely limit ourselves to deterministic models. Most of the key intuitions we get from the deterministic case also apply to probabilistic voting,⁵

⁴ Exactly how we generalize this probability function for the case where the voter is choosing from among more than two alternatives without breaking the choice down into a sequence of pairwise choices is beyond the scope of the present essay. The most common way to deal with this problem is in terms of *Luce's Choice Axiom* (Luce, 1959); for a more general discussion see Coughlin (1984, 1992).

⁵ For example, the results we give below linking transitivity to proximity voting over a unidimensional continuum can be reformulated for probabilistic voting in terms of *stochastic transitivity*. There are, however, some differences in results when we model party competition probabilistically rather than deterministically whose explication would take us beyond the scope of this introductory essay.

II. Politics in One Dimension

While clearly there are important instances where we need to take into account multiple issue dimensions, especially when we look at coalition politics in multi-party democracies (see e.g., Schofield, this Handbook), and also valence issues that include candidate characteristics such as trustworthiness and personal integrity (see e.g., the 2016 U.S. presidential election), if we look to how political competition is structured at the aggregate and elite level, for many purposes we may think of politics as occurring in a predominantly one dimensional world. Poole and Rosenthal (2011), for example, find more than eighty percent of the voting patterns in the U.S. House and Senate by individual Representatives and Senators can be explained by acting as if bill locations and the ideal points of those who vote on them were embedded in a single commonly perceived dimension.⁶ But the importance of the second dimension has been recently shrinking, so that unidimensionality and very high levels of party polarization on just about everything has become the norm.

Some explanations for this empirical regularity focus on the nature of the party system. Hinich and Munger (1996) posit that *dimensional reduction* occurs because, regardless of how many different issues there may actually be, the choice over any issue can be viewed as taking place only among the projections of that issue onto the line or plane or hyperplane defined by the platforms of the various parties. Thus, from this perspective, two party competition in the U.S., “naturally” falls into a unidimensional form of issue contestation, since two points define a line. Similarly, three party politics requires a two dimensional representation, etc. Earlier work, Taagepera and Grofman (1985), following Lijphart (1984), made a similar point. They note that, empirically, the number of issue dimensions in party competition is related to the number of parties by the equation $I = N - 1$, where I is the number of issue dimensions and N is the (effective) number of political parties.⁷ In this form of the equation, issue dimensionality is

⁶ However, the second dimension is still important because it provides the basis for much of the crossover voting that at least partly cuts across party lines.

⁷ The *effective number of political parties* (Laakso and Taagepera, 1979), commonly abbreviated as the L-T index, is a mathematical construct intended to take into account the fact that not all parties are of the same size. The *effective number of political parties in the electorate* is defined as the inverse of the sum of squared party vote shares. The *effective number of*

being explained by the number of parties, which in turn needs to be explained by other factors.⁸ However, Taagepera and Grofman (1985) also point that this equation can be written as $N = I + 1$, suggesting reciprocal causality.

We should not think that one-dimensional political competition is only important in two party systems. For many democracies with multiple parties the empirical claim is made that there is primarily a single dimension, usually but not always a left-right dimension defined in terms of the size and scope of government that has been the main feature of post-WWII political competition⁹. Glazer and Grofman (1988) offer an explanation for the restriction of politics to a space of low dimensionality that is related to those discussed above. They argue that it is easiest for politicians to couch arguments in an ideological fashion by simplifying the exposition to combine multiple issues, rather like putting on a pre-knotted tie. Yet another reason why issues tend to be bundled has to do with the role of interest groups. In seeking to put together winning coalitions, even interest groups that are largely single-issue will look for allies who may be willing to endorse their position in return for support on issues that the other group ranks of greater importance. The creation of such linked patterns of reciprocity can lead to small blocs of allied interests -- in the limit, to polarization into two blocs under the norm that the friend of my friend is my friend and the enemy of my friend is my enemy. Because multiple issues are projected onto a lower dimensional surface, we expect to find that issues tend to become conflated. For example, a position “on the left” may come to include support for gay marriage and support for a relatively open immigration policy, and not just support for a major role of government in the economy, even though, in principle, one need not imagine that positions on these issues are logically interrelated.

political parties in the parliament is defined as the inverse of the sum of squared party legislative seat shares. Note that the L-T index is simply the inverse of the Hirschman-Herfindahl index of fragmentation.

⁸ See e.g. Grofman, Bowler and Blais (2008) for attempts to explain why countries do or do not have two-party politics.

⁹ For example, the Comparative Study of Electoral Systems (CSES) Project, a major cross-national data set, now with multiple waves, codes data on party locations in unidimensional terms.

Of course, how elites and the well-informed see the space of political competition may not capture the perceptions of ordinary citizens. Perhaps most ordinary citizens do not see political choices in the same structured way as more sophisticated voters, e.g., they are more often muddled. On the other hand, we can also imagine that some voters make their choices based on one or at most a handful of issues that are more or less orthogonal to the one-dimensional structure that defines elite political competition. As we show later, however, as long as we are interested less in individual decision making than in how individual preferences aggregate into collective preferences, especially in terms of majority preferences, even when many voters do not see the world in unidimensional terms, majority rule processes may nonetheless be interpreted as occurring “as if” the choice space was (for most practical purposes) unidimensional.¹⁰

Some further definitions and propositions are useful.

DEFINITION: An alternative is said to be a *majority winner* (a.k.a. *Condorcet winner*) if it is majority preferred to each and every other alternative in paired completion.

DEFINITION: Let $m(A, B)$ denote the *majority preference relation* of A versus B, taking on the value 1 if A is majority preferred to B, and -1 if B is majority preferred to A.

DEFINITION: Majority references over a set of alternatives are said to be *transitive* if, for any triple of alternatives $\{A, B, C\}$, $m(A, B) = 1$ and $m(B, C) = 1$ implies $m(A, C) = 1$.

DEFINITION: Majority references over a set of alternatives are said to contain a *majority preference cycle* if, for some triple of alternatives $\{A, B, C\}$, subject to relabeling

$$m(A, B) = 1, \quad m(B, C) = 1, \quad \text{but } m(A, C) = -1$$

For a given set of real world preferences, we cannot expect to always find a majority winner, still less to always find transitive preferences. Indeed, there is a huge literature seeking to determine the likelihood of majority preference cycles and the conditions for transitivity (for seminal contributions see e.g., Arrow 1951, 1962; Plott, 1967; Sen and Pattanaik, 1969; Sen,

¹⁰ Nonetheless, when we do empirical tests of the predictive accuracy of a spatial model of politics, the fact that some voters are uncertain about candidate and party locations, or even uncertain about their own ideological predispositions, introduces error into the estimates (see Serra, 2016)

1970; Saari, 1994, 1995). However, the existence of a unidimensional continuum along which both voter ideal points and alternatives are arrayed, coupled with deterministic voting, guarantees the existence of both a majority winner and a transitive majority preference ordering.

THEOREM (Black, 1958): In deterministic proximity voting along a single dimension, the alternative which lies closest to the *median voter* (i.e., the voter such that exactly half of the voters lie to the right of that voter and half to the left of that voter)¹¹ is a majority winner.

THEOREM: Let M denote the alternative that lies closest to the ideal point of the median voter. In deterministic proximity voting along a single dimension,

if $d(A, M) < d(B, M)$ then $m(A, B) = 1$.

THEOREM (Black, 1958): In deterministic proximity voting along a single dimension, if we rank order alternatives in terms of their proximity to the ideal point of the median voter, this proximity ordering generates a transitive majority preference ordering among the alternatives.

Single-peaked Preferences

DEFINITION: By a voter's *utility function* we mean a function that allows us to specify the (relative) value assigned by that voter to each of the feasible alternatives.

DEFINITION: For a given ordering of alternatives along a unidimensional continuum, that ordering is said to be *single-peaked* with respect to that ordering if, for each voter, the utility function of that voter, when plotted, generates a curve with no more than a single inflection point, from up to down.

THEOREM: Deterministic proximity voting along a single dimension generates a utility function in terms of distance that can be represented in terms of single-peaked curves.¹²

¹¹ Recall that we have assumed away the possibility that there is more than one voter located at the same point on the line, and we have posited an odd number of voters.

¹² Proximity based deterministic decision-making over a single dimension implies single-peakedness, but not conversely since a curve may be single-peaked without utility dropping

It is informative to relate the notion of single-peakedness to *voter preference profiles*.

For the case of three alternatives {A, B, C}, it should be apparent that, when all voters have preferences that are single-peaked wrt to some ordering of the three alternatives, only four of the six possible strict linear preferences among the three alternatives can be found. See Figure 1. Imagine, for example, that alternatives are embedded on a line with A as the leftmost alternative and C as the rightmost alternative. Now, clearly, only ABC, BCA, BAC, and CBA generate single-peaked curves. And similarly, for any other location of alternatives on the line we might propose, there will be only four orderings that are feasible. It should also be apparent that what defines this set of four rankings is that in none of them has B, the median alternative wrt to the specified line, last. Thus we can think of single-peakedness as a *domain restriction* in the sense of Arrow (1962), and indeed, Arrow (1962) contains a “Possibility Theorem for Single-Peaked Preferences.”¹³

<<Figure 1 about here>>

Domain Restrictions and Net Preferences

Single-peakedness, which Sen (1970) refers to as the *not-worst condition* (NW), i.e., a situation where there is an alternative which is never found at the bottom of any voter’s preference ranking, is only one of a set of domain restrictions on sets of three alternatives sufficient to guarantee the absence of majority rule cycles. The two others are what Black (1958) refers to as *single troughedness*, which can be given a clear geometric interpretation, with curves with at most one inflection point from down to up -- a condition labeled by Sen (1970) as the *not-best condition* (NB), i.e., a situation where there is an alternative which is never found at the top of any voter’s preference ranking; and a further parallel condition *not-middle* (NM), which lacks such a clear geometric interpretation. The NM condition can, however, be given a kind of intuitive interpretation in terms of willingness to compromise. Imagine that a society is choosing

symmetrically with distance from the voter’s ideal point, e.g., for some voters, utility may drop more precipitously as we move left (right) of the voter’s ideal point.

¹³ For a much more thorough discussion of Arrow’s Theorem and how it relates to cyclic majorities see the Saari chapter in this volume.

between option A, involving a full scale military invasion of a country in the Middle East headed by a very nasty dictator directed at overthrowing that dictator; with option B that of providing some military support to (supposedly) pro-Democratic forces in the country, but refusing any commitment to put major military forces on the ground; and option C being stay out entirely and let the locals settle things among themselves (perhaps also with the involvement of some external players seeking to prop up the dictator). If there are only two kinds of voters: one group, whom we might call *compromisers*, who put the compromise option, B, as their first choice; and the other group, the *anti-compromisers*, who put the compromise option, B, as their last choice, and if members of each group have both hawks (who rank A over C), and doves in their ranks (who rank C over A), then we get four preference rankings that satisfying NM.

THEOREM (Sen and Pattanaik, 1969; Sen, 1970): If, for every set of three alternatives, the set of orderings among those alternatives satisfy either the NW, or the NB or the NM condition; then there is a transitive majority preference ordering.

This set of conditions is commonly referred to as the *value restriction condition*.

Only if all orderings satisfy *value restriction* can we guarantee that preferences will be transitive. In other words, the value restriction theorem is a kind of impossibility theorem, about what cannot be ruled out if all possible preference orderings are feasible i.e., it is about what can or cannot happen regardless of the number of voters who hold particular preference orderings. For the case of three alternatives, all three domain restricting conditions require that all voters are restricted to only four of the six possible strict preferences. But it is not just any set of four preference orderings out of six that will do. Saari (1994, 1995; see also Saari, 2014) shows that we obtain transitivity if, for every triple of alternatives, the set of preference profiles excludes at least one ranking from each of the two possible preference cycles: the one generated by abc, bca and cab, and the one generated by acb, bac, and cba.

Unfortunately, because of the language Sen himself uses, it is a common error to believe that, the only way in which we could avoid cycles and guarantee transitivity for some set of voter rankings of three alternatives is to restrict preferences to four out of the six possible strict orderings. That claim is false. Consider nine voters with preferences ABC, ACB, ACB, BAC, BCA, BCA, CAB, CBA, and CBA. Clearly, all six strict linear preferences are found and yet majority rule displays the transitive ordering CBA. Similarly, if there is a single preference that

commands a majority of the preferences among three alternatives, then majority rule preferences are transitive.

To understand how we can have acyclicity in without satisfying NB or NW or NM, the mathematical sociologist Scott L. Feld developed the idea of *net preferences* (Feld and Grofman, 1986).¹⁴ Let $n(UVW)$ be the number of voters with the preference UVW . Let the *net preferences* of any given ordered triple, say UVW , be denoted $NP(UVW)$, where

$$NP(UVW) = n(UVW) - n(WVU) \text{ if } n(UVW) > n(WVU), \text{ and } 0 \text{ otherwise.}$$

It should be apparent that, for three alternatives, the maximum number of orderings that can have *positive net preferences* is three.

THEOREM (Feld and Grofman, 1986): If, for every set of three alternatives, the set of orderings with *positive net preferences* satisfy either the NW, or the NB or the NM condition; or if there is a single net preference that commands a majority of the net preferences among those three alternatives, then there is a transitive majority preference ordering.

The set of conditions in the above theorem is referred to as *net value restriction*. To return to the nine voter example above, the net preferences are ACB , BCA , and CBA , with $n(ACB) = n(BCA) = n(CBA) = 1$. Among these net preferences C is never worst, and thus *single-peakedness* is satisfied.¹⁵ But, if we have a single net preference ranking that commands a majority we can also provide transitivity, regardless of the other voter's preferences and of how many distinct orderings are present.

The Feld-Grofman theorem is potentially very important in understanding how politics can be one-dimensional despite the presence of voters whose preferences are not based on the same single-peaked ordering. As Feld and Grofman (1986) point out, preferences can, in net terms, satisfy the NW condition even if not all voters, or even not many voters, have single-

¹⁴ They later realized that this idea had been previously proposed by Gaertner and Heinecke (1978) and is discussed, though in a somewhat cryptic fashion, in Sen (1970). More recent (and more general) uses of the net preferences concept are found in Regenwetter et al. (2006) and references cited therein.

¹⁵ Also, A is never in the middle, and thus the NM condition (also called the *polarization condition*) is also satisfied.

peaked preferences. This can occur, for example, when voter preference ordering reflect media or elite discussion of the location of alternatives, but they do not do so perfectly. If we think of preference formation as based on voters who do know their own ideal point on an ideological continuum but who have difficulty assessing the location of the alternatives on that continuum i.e., their perceptions are contaminated by noise then, in the aggregate, preferences can be single-peaked even though a number of voters lack single-peaked preferences because they have wrongly placed alternatives on the continuum vis-a-vis one another. Even though mistakes are made that create violations of single-peakedness, the *net* result of such mistakes are apt to cancel out.

Another way in which preferences can, in net terms, satisfy the NW condition even if not all voters, or even not many voters, have single-peaked preferences is when there is a subset of knowledgeable voters who are fully cognizant of the true (unidimensional) ordering of alternatives and rank alternatives accordingly, and the rest of the electorate is completely ignorant about this ordering and thus making choices that appear completely random wrt to the dimension on which single-peakedness is defined. Here, if there is enough randomness, the “signal” (the knowledgeable voters whose preferences are single-peaked) can swamp the “noise” (the uninformed voters). Yet another, related, way to get single-peakedness is if voters differ in their ability to distinguish different subsets of ideologically proximate alternatives, e.g., perhaps conservative (left-leaning) voters are able to make fine distinctions among conservative (left-leaning) candidates but tend to lump more left-wing (right-wing) candidates into a catch-all “left” (“right”) category or make mistakes as to who is further to the left (right). But when we combine input from voters of different ideological leanings, each with specialized more accurate knowledge, the overall continuum may be quite accurately reproduced if we look at majority rule preferences rather than belabor the classificatory “mistakes” of individual voters.

Above we have emphasized how what may be true for the electorate as a whole may not true of all or even most voters. This fact has important implications for empirical analyses. In particular, countries differ in what proportion of the public can locate themselves on a left right continuum, and even those citizens who claim they can do so may be making mistakes in where they place parties. So, at the individual level for citizens (as opposed to elites, such as legislators) there is a practical ceiling as to how well a simple unidimensional Downsian model can be

expected to work if our goal is to predict how individual voters vote. And, even if voters are able to locate both themselves and parties on a single dimension, this is far from guaranteeing that voters will vote for the closest party on that dimension. (a) There can be strategic voting, with voters eschewing votes for parties seen as having little chance of winning, (b) there can be valence concerns like party loyalty and performance evaluation playing a major role in determining voter choice, (c) voters can be concerned about influencing not just the present election outcome but also future behavior by political parties, e.g., voters may wish to send a message by voting for a party more extreme than they actually want in order to move their preferred party in that ideological direction and, relatedly, (d) there can be directional as well as proximity voting (see discussion below). Finally, and perhaps most importantly, (e) if there are multiple issue dimensions of evaluation, there can be differing salience on those different dimensions with, for example, single issue voters behaving in ways that may not make sense in left-right terms. (Downs, 1957).

However, while there are many reasons to explain divergence among how voters evaluate alternatives, this does not really affect the Feld-Grofman (1986) demonstration that a unidimensional pattern can result when we examine not individual voters but the overall majority rule preference relation. The extent of unidimensionality at this aggregate (majority rule) level is a matter for empirical investigation (see e.g., Feld and Grofman 1988).

The Single Crossing Condition

NB and NW can also be given a geometric interpretation in the context of binary preferences organized in what we might call a “pseudo unidimensional” way. This representation, in terms of what is called the *single-crossing condition*, is based not on voter utility functions defined over voters on a line but, rather, on the linear representability of cut points between pairs of alternatives in terms of voter preferences between those pairs.

DEFINITION: A set of voter (strict linear) rankings is said to satisfy the *single crossing condition* if wrt to some line, all preferences ranking any given alternative above some other

given alternative all lie to one side of the line in the space of preference rankings from those that have the reverse ranking on the pair, and the cut points do not cross or intersect.¹⁶

This definition is still rather abstract and can best be explained with examples. We will explicate the *single-crossing condition* for the case of three alternatives. Figure 2 is a graph showing a three alternative example with voter preferences that satisfy *single crossing*.

<<Figure 2 about here>>

The reader will have noticed that, as shown in Figure 2, the way in which single crossing works generates segments of the line which correspond to particular voter preference orderings. The alert reader will also have noticed that, with three alternatives: (a) there are only four of the six possible strict preference rankings identified as feasible, and (b) the set of ordering shown in Figure 3a are ones that satisfy the NW condition in that y is never the last choice in any of the rankings.

While *single crossing* has been linked to *single-peakedness*, it is usually portrayed as distinct from other Sen-Pattanak type domain restrictions, but that is not really accurate. Figure 3, taken from Saporiti (2009: 134) illustrates how, for three alternatives, *single-troughed* (NB) preferences can also generate a set of ordered pairs that satisfy the *single crossing* condition.

<<Figure 3 about here>

THEOREM (Roberts, 1977; Grandmont, 1978): If, for a given set of three alternatives, pairwise cutpoints can be ordered in a way that satisfies the *single crossing condition*, then majority rule preferences are transitive.

However, the converse to the theorem does not apply. As is well known (see e.g., Rothstein 1990, 1991), transitivity does not imply single crossing. While our examples showing that NW and NB preferences over three alternatives can be arrayed so as to satisfy single-crossing have already shown that preferences that always generate transitive orderings in either

¹⁶ We take this definition from Saporiti (2009) with some slight changes in wording.

of these ways can be arrayed to satisfy the single crossing condition,¹⁷ the same is not true for preferences that satisfy the NM condition (and are thus necessarily transitive). Grofman and Feld (2016) show that, for any triple of alternatives, the assumption that preferences are both NM and satisfy single crossing leads to a contradiction except when there are only three preference rankings with nonzero weight and these are non-cyclic. In such a case, the set of three rankings either is either NM and NW or NM and NB.¹⁸

When preferences satisfy *single crossing* there is a way of ordering feasible preference rankings from left to right, and thus we can order voters in terms of those preference ranking from left to right, so that we can then find the preference ordering that corresponds to the preference of the median voter wrt to this ordering.

THEOREM (Roberts, 1977; Grandmont, 1978): When we order the pairwise cutpoints that satisfy the single crossing condition from left to right on a line, and arrange the set of voter orderings accordingly, the median voter preference is the one that defines the majority preference ordering for the society.

¹⁷ The particular domain restriction that generates transitivity for some given set of three alternatives may be different from the domain restriction that other triples of alternatives satisfy. Consider an example taken from Saporiti (2009), with three feasible preferences orderings over four alternatives: $xyzw$, $zyxw$, $yxwz$. There is a transitive majority preference ordering, the nature of which can vary with the exact distribution of preferences. If we have one voter with each of these three preferences the ordering is $yxzw$. However, the set of three alternatives $\{x, y, z\}$ is single-peaked (NW) with respect to the line oriented xyz , where y can be regarded as the median choice on that line. In contrast, any set of three alternatives with w in it satisfies the NB condition. For a set of multi-alternative preference rankings satisfying the single crossing condition, a natural question is under what condition will the same domain restriction apply to all triples. One answer is that all triples will be single-peaked when voter preferences are Euclidean over alternatives located on a line. (Elkind, Faliszewski, Skowron, 2014). Preference orderings that satisfy both single-peakedness and the single crossing condition are called SPSC

¹⁸ Donald Saari (personal communication, November 11, 2016) subsequently pointed out that the incompatibility of NM and *single crossing* can be derived as a corollary of results in Saari (2014).

Of course, since transitivity under *single-crossing* is provided by either the NB or the NW condition, the above result follows straightforwardly from the existence of *single-peaked* or *single-troughed* preferences. It is when we have a mix of both conditions that the theorem has some bite. Rothstein (1991) calls this result the *Representative Voter Theorem* (RVT), and we may think of it as having Black's median voter theorem as a special case.

Thinking about conditions on preferences in terms of *single crossing* can yield insights that we might miss by the usual approach in terms of *single-peakedness* or *single-troughedness*. For example, the *single crossing condition* has been used by economists in various substantive domains beginning with the classic work by (Meltzer and Richard, 1981, 1983) on income redistribution and the collective choice of a tax rate. A key insight of this work is summarized in Ashworth and Mesquita (2006, pp. 217–218):

Suppose a moderately rich individual prefers a high tax rate to another relatively smaller tax rate, so that he reveals a preference for a greater redistribution of income. Then, the single-crossing property requires that a relatively poorer individual, who receives a higher benefit from redistribution, also prefers the higher tax rate.¹⁹

Alternatives to Pure Proximity-Based Voting

While the focus of this essay is proximity voting, there is another important class of models of spatial voting, *directional models* that we wish to discuss briefly. Such models usually require us to identify the location of the *status quo* in the multidimensional space. In this class of models the status quo serves as a kind of baseline referent. In particular, *directional models* posit that individuals may prefer (some) alternatives that are located in the same direction from the status quo (or some central “neutral” point) as is the voter's ideal point to ones that are on the other side of the status quo point (Cohen and Matthew, 1980; Rabinowitz and McDonald, 1989), even if the latter are closer in proximity terms. For the unidimensional case, for a choice between two alternatives, A and B, we show an example of such a potential conflict between directional and proximity voting in Figure 3. In this and succeeding figures, alternatives are

¹⁹ They also note that, sometimes, this is interpreted in the literature as implying that “there is a complementarity between income and taxation, in the sense that lower incomes increase the incremental benefit of greater tax rates.” We identify a few other uses of the single crossing idea in the companion chapter to this one on applications of unidimensional models.

represented by capital letters, and the voter ideal points are designated by lower case letters. We use sq to represent the status quo. We illustrate directional models only for the case of unidimensional preferences.

<<Figure 4a about here>>

In Figure 4a, we see that A is closer to i than is B, but B is on the same side of the status quo as is i. The intuition underlying the directional model of voting is that policy changes to any given status quo are rarely fully implemented since there are inertial forces that may retard change. In this context, a move in the direction of A necessarily takes the status quo further away from voter i; while a move in the direction of B necessarily results in the status quo moving in voter i's direction, albeit with a possibility of overshooting voter i's most preferred position. Thus, there may be circumstances where the gamble balancing the possibility of gain against the possibility of still greater loss if voter i chooses B may be preferred to the certain level of loss if A is chosen.

When we posit directional voting we must specify the conditions under which an alternative on the same side of the status quo as the voter's ideal point will be preferred by the voter to one on the opposite side of the voter's ideal point. There have been several different ways to address that question.

In the original version of the Rabinowitz and McDonald formulation of directional voting, there is an arbitrary *zone of feasibility*, such that alternatives outside that zone are never chosen. We can illustrate this idea in Figure 4b. Here F will never be chosen.

<<Figure 4b about here>>

Grofman (1985b) proposed to combine directional and proximity voting by adding a further *discounting parameter* to reflect the "realistic" potential for change from the status quo. In the Grofman model, the distance between a proposed bill or party/candidate platform is discounted by a factor $1 - r$, where $0 < r < 1$, so that, if the actual distance between voter i's ideal point and the status quo point is $d(i, sq)$, the voter acts as if that distance is only $(1 - r) * d(i, sq)$. Now we can again have a situation in which the voter prefers an option further away from her

ideal point to one that is closer.²⁰ This case is illustrated in Figure 5. We posit that r is high, thus no matter that B and C propose to make dramatic changes from the status quo, i takes the expected change to be quite small. We show i 's expectations about the change from the status quo that will occur as A, B, or C, is the policy platform chosen, as the points A', B and C'.

<<Figure 5 about here>>

Grofman (1985b) proposes the discounting model as a way of explaining why there can be change in voter choices without any apparent change in voter preferences. In the usual proximity model preferences are determined by distance between the voter's ideal point and each of the various alternatives. But, if the status quo moves, as a result, say, of implementation of new policies by a governing party, then, under the discounting model, some voters who voted for that party previously, will now find themselves with changed preferences even if the distance between their ideal point and the various party platforms remains the same. We illustrate one way for this to occur in Figure 6.

<<Figure 6 about here>>

Initially, the discount factor, r , say .6, is such that B is chosen over A by voter i . If voter i is pivotal, then the point halfway between sq^{old} and B becomes the expected location of sq^{new} . But now the new status quo has "overshot" voter i 's ideal point by moving too far to the right. Now, with the same discount factor of .4, voter i prefers A over B, and A over C, rather than, as before, C over B, and both over A. If voter i is pivotal, because the status quo has shifted, we can observe an alternation in power even though the location of pivotal voter i 's ideal point has not moved, and the policy platforms of the parties remain the same in both elections.

²⁰ For simplicity we may assume this parameter to be the same for all proposals; however, if we are in the party competition context, we might think that there would be a different discounting factor for the policy promises of incumbents and those of challengers.

Note, however, that we have implicitly assumed that discounting is the same for both candidates. It could well be that discounting applies particularly to challengers, who are more of an unknown quantity.²¹

A third approach to combining directional and proximity considerations is the “unified model” proposed by Samuel Merrill (see e.g., Merrill and Grofman, 1997, 1999; cf. Adams, Merrill, Grofman, 2005). In this model there is a “mixing parameter,” β , which represents the relative weight being given by voters to proximity and directional aspects of the choices open to them. The value of this parameter for the “average” voter is estimated empirically from survey data by MLE methods. Estimates of this parameter vary widely across countries and sometime even within-country across elections, but usually the estimate is such that we would believe that both proximity and directional factors matter. Intuitively we may see that, the greater the weight given to the directional component in the voters’ calculus, the greater the likelihood that ideologically extreme parties will gain representation.²² Under the mixed model, voters no longer necessarily vote for the party or candidate closer to them.

In various work by Merrill and/or Adams and/or Grofman, (see esp. Adams, Merrill and Grofman, 2005; Merrill, Adams, Grofman, forthcoming), a third complication is introduced into the spatial model, *party loyalty*. When voters make choices at least in part based on party loyalty, voters again do not necessarily vote for the party or candidate closest to them ideologically -- unless that candidate is “much” closer than the candidate of the party to which they have some level of loyalty. When voters' partisan loyalties influence their voting decisions, and these allegiances correlate with their policy preferences, then parties are motivated to locate closer to their own loyalists than would be the case in the standard Downsian model without party loyalty.

In the next sections we turn from models in which voters pick a single winner based on voters ranking alternatives in terms of proximity to their most preferred outcome to other types

²¹In the companion chapter on applications, we briefly discuss the application of the discounting idea to support for Donald Trump in 2016.

²²Grofman’s discounting model can, in part, be thought of as formalizing some insights found in a classic essay by Donald Stokes (1963), and may also be viewed as a precursor to the Merrill “unified model.”

of models. In these new models, voters can still be characterized as points on a line, but the nature of the line may be somewhat different, e.g., the line may represent an ordering of voter “thresholds,” or the line may represent an ordering of voter “competence,” in which outcomes can, in principle, be ranked from best to worst, but voters can be seen as differing in some (probabilistic) scale of evaluative expertise in distinguishing the best from the worst. And, in some of the models we present below, the outcome is not choice of a single alternative but of a subset of alternatives.

Threshold Models

Perhaps the simplest kind of threshold model on a line is one in which every voter has a threshold that corresponds to a point on the line, and there is a signal that can be viewed as a point on the line triggering a dichotomous YES-NO choice. Voters whose threshold is on the left side of the point vote YES, and voters who are the right side of the point vote NO. Here, we may establish a collective preference simply by tallying YES and NO votes. If the median voter is to the right of the signal, then the proposal fails. This signal detection-threshold model can be applied to study turnout variation across both individuals and elections.

A second threshold model can be applied to what is commonly called *subset choice*, where voters select a subset of alternatives as “acceptable.” Now we must combine the subsets selected by each voter to determine a collectively acceptable/preferred set --which may consist of a single alternative, or may consist of multiple alternatives. In the continuous version of this model voters select one or more subsets of a line as “acceptable,” and we again seek to identify a collectively acceptable or preferred domain, consisting of one or more subsets of the line. One way to achieve this collective consensus is to look for alternatives (or line segments) that are located in the acceptance set of a majority of the voters, and to change the status quo only if there is such majority agreement. Another way is to use *approval voting* (Brams and Fishburn, 1983), by assigning to each choice/each line segment a vote tally given by the number of voters who approve of it, and then selecting the alternative /set of alternatives with the highest approval. If we posit distance-based preferences, we can use the idea of single-peaked preferences to put restrictions on voter’s allowable subset choices.

Another application of thresholds is one where there are two or more groups, each with a distribution of values for its members, where the values can be thought of as points on a line. If we set a threshold, say for what is required to pass some exam, the first question to be answered is “What proportion of each group has values exceeding the threshold?” If, for simplicity we assume that each group’s values can be characterized as a normal distribution with a given mean and variance, we can use well-known statistical tools to answer this question. Now, imagine that we have two groups, A and B. The second question to be answered is “How does the ratio of the proportion of group A’s success rate to the success rate of group B change as we increase the threshold?” Because we are dealing now with ratios of tails of normal distribution it turns out that even groups that differ little in their mean characteristics may have very very different success rates when the bar for success is set very high; similarly, even groups with identical means but different variances may differ substantially in relative success as we increase the bar for success. The extent of these differences can be very surprising even to those familiar with properties of the normal distribution (Grofman and Merrill, 2004).

Judgmental Models

The distinction between choices based on “preferences” versus choices based on “judgments” (sometime called *epistemic judgements*) has become a common one in the literature on democratic theory (Grofman, Owen and Feld, 1983; Cohen, 1986; Coleman and Ferejohn, 1986; Grofman and Feld, 1988; List and Goodin, 2001; Giere, 2002). In preference models, voters rank alternatives in terms of proximity to their most preferred outcome, with voters differing among themselves in which outcome they regard as best for themselves. In unidimensional judgmental models, voters are assumed to have a certain likelihood of making a “correct” judgment, or there is some unidimensional scale on which voters differ, where high values indicate a higher probability of taking a certain action.²³

²³ In unpublished work, Feld and Grofman (2016) provide an intuitive statistical test for when a preference profile exhibits judgmental as opposed to ideological/preference-based characteristics.

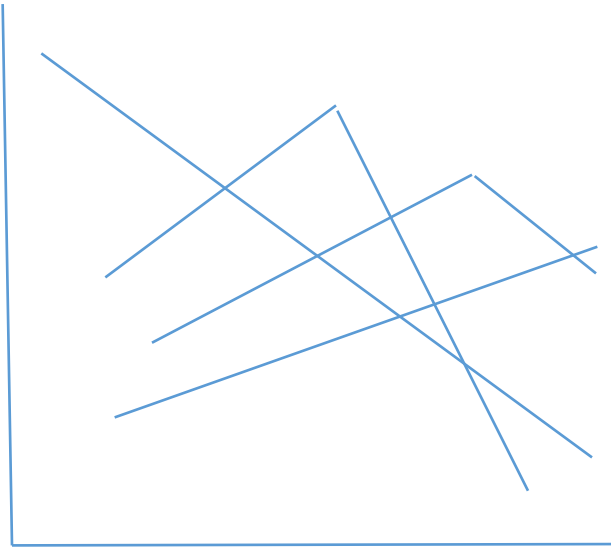
For binary choices, when we aggregate judgments of voters with different competences, there are two key results. The first is the *Condorcet Jury Theorem*, which says that when the (mean) competence of the group is above .5, the probability that the group majority will make the better of the two choices increases with (mean) competence, and approaches one as the group size increases, but goes toward zero with increasing group size if (mean) competence is below .5.²⁴ The second, the *Nitzan-Paroush-Shapley-Grofman Theorem*, says that when individuals differ in their judgmental competences and there is disagreement within the group as to which is the better of two choices, *ceteris paribus*, make the choice such that the sum of the log odds of voter competences on the one side of the issue exceed the sum of the log odds of the voter competences on the other side of the issue. This result is independently discovered by Nitzan and Paroush (1983; see also Nitzan and Paroush, 1985) and by Shapley and Grofman (1984). It turns out, however, to be a form of *Bayes' Theorem* in disguise. A discussion of judgmental models is found in the Nitzan and Paroush chapter in this volume and we will not discuss them further here, or in the applications chapter that is the follow-up to this one.

²⁴ The progenitor of the “choices as judgments” approach is Condorcet (1785). Condorcet’s essay was most prominently brought to the attention of economists and political scientists by Duncan Black (1958).

III. Discussion

The spatial model is both remarkably flexible and very broad in its scope. Even when it does not tell the whole story, it usually captures an important explanatory factor. And, when we simplify the spatial model to posit unidimensionality we can generate clear and testable empirical claims. In the companion chapter to this one we consider applications of the spatial model in the context of one dimensional politics in areas such as legislative voting, party competition, and coalition formation.

Figure 1: Three Alternative Example of Single-Peaked (NW) Preferences



Utility

Figure 2 : Three Alternative NW Example Showing Voter Preferences that Satisfy the Single-Crossing Condition

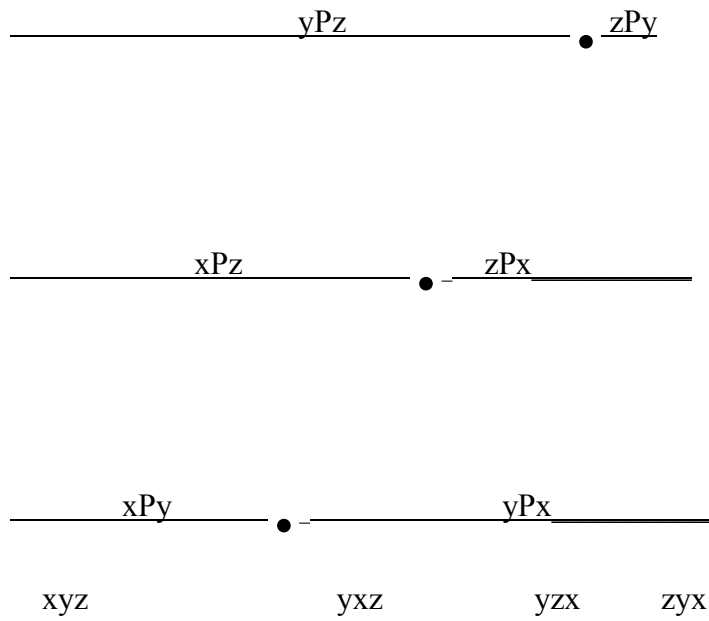


Figure 3: Three Alternative NB Example (Saporiti, 2009: 134) Showing Voter Preferences that Satisfy the Single Crossing Condition

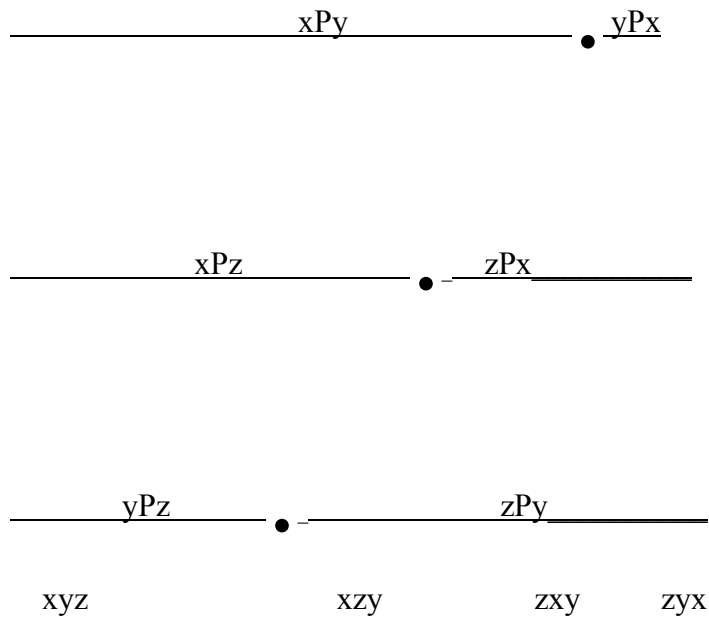


Figure 4a: Unidimensional Example Showing Location of Status Quo Point

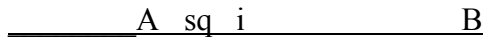


Figure 4b: Pairwise Choice Between A and F, with F Outside the Feasible Zone: The Choice for Voter i Dictated by Proximity and the Choice Dictated by Direction from the Status Quo are Now the Same



Figure 5: Actual Platforms $\{A, B, C\}$ and Highly Discounted Platforms $\{A', B', C'\}$: Under the Grofman Discounting Model, Voter i is closer to C' than to either A' or B' even though both A and B are closer to voter i 's ideal point than is C .

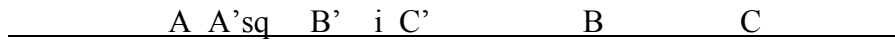
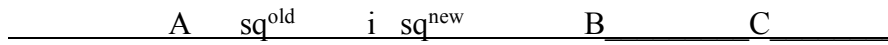


Figure 6 : Under the Grofman (1985) Discounting Model, the Preference of Voter i for C over A (and for B over A) Changes When C is Chosen and, as result, the Status Quo Subsequently Shifts from sq^{old} to sq^{new} Even Though Nothing Else Has Changed



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